meaningfulness and invariance. Few disavow the principle that scientific propositions should be meaningful in the sense of asserting something that is verifiable or falsifiable about the qualitative, empirical situation under discussion. What makes this principle tricky to apply in practice is that much of what is said is formulated not as simple assertions about empirical events – such as a certain object sinks when placed in water – but as laws formulated in rather abstract, often mathematical, terms. It is not always apparent exactly what class of qualitative observations corresponds to such (often numerical) laws. Theories of meaningfulness are methods for investigating such matters, and invariance concepts are their primary tools.

The problem of meaningfulness, which has been around since the inception of mathematical science in ancient times, has proved to be difficult and subtle; even today it has not been satisfactorily resolved. This entry surveys some of the current ideas about it and illustrates, through examples, some of its uses. The presentation requires some elementary technical concepts of measurement theory (such as representation, scale type, etc.), which are explained in MEASUREMENT, THEORY OF.

INTUITIVE FORMULATION AND EXAMPLES

The following example, taken from Suppes and Zinnes (1963), nicely illustrates part of the problem in a very elementary way. Which of the following four sentences are meaningful?

(1) Stendhal weighed 150 on 2 September 1839.

(2) The ratio of Stendhal's weight to Jane Austen's on 3 July 1814 was 1.42.

(3) The ratio of the maximum temperature today to the maximum temperature yesterday is 1.10.

(4) The ratio of the difference between today's and yesterday's maximum temperature to the difference between today's and tomorrow's maximum temperature will be 0.95.

Suppose that weight is measured in terms of the ratio scale $\mathscr{W}$ (which includes among its representations the pound and kilogram representations and all those obtained by just a change of unit) and that temperature is measured by the interval scale $\mathscr{T}$ (which includes the Fahrenheit and Celsius representations). Then Statement (2) is meaningful, since with respect to each representation in $\mathscr{W}$ it says the same thing, i.e., its truth value is the same no matter which representation in $\mathscr{W}$ is used to measure weight. That is not true for Statement (1), because (1) is true for exactly one representation in $\mathscr{W}$ and false for all of the rest. Thus we say that (1) is 'meaningless'. Similarly, (4) is meaningful with respect to $\mathscr{T}$, but (3) is not.

The somewhat intuitive concept of meaningfulness suggested by these examples is usually stated as follows: Suppose a qualitative or empirical attribute is measured by a scale $\mathscr{S}$. Then a numerical statement involving values of the representation is said to be *meaningful* if and only if its truth (or falsity) is constant no matter which representation in $\mathscr{S}$ is used to assign numbers to the attribute. There are obvious formal difficulties with this definition, for example the concept of 'numerical statement' is not a precise one. More seriously, it is unclear under what conditions this is the 'right definition' of meaningfulness, for it does not always lead to correct results in some well-understood and non-controversial situations. Nevertheless, it is the concept most frequently employed in the literature, and invoking it often provides insight into the correct way of handling a quantitative situation – as the following still elementary but somewhat less obvious example shows.

Consider a situation where M persons rate N objects (e.g. M judges judging N contestants in a sporting event). For simplicity, assume person i rates objects according to the ratio scale $\mathscr{R}_i$. The problem is to find an ordering on the N objects that aggregates in a reasonable way the persons' judgements. It will be assumed that their judgements cannot be coordinated in such a way that, for R_i in $\mathscr{R}_i$ and R_j in $\mathscr{R}_j$, meaning can be given to the assertion $R_i = R_j$. (The difficulties underlying such a coordination are essentially those that arise in attempting to compare individual utility functions. The latter problem – 'the interpersonal comparison of utilities' – has been much discussed in the literature, as for example in Narens and Luce (1983) and Sen (1979). It is generally conceived that there are great, if not insurmountable, difficulties in carrying out such comparisons.) Any rule that does not involve coordination can be formulated as follows: First, it is a function F that assigns to an object the value $F(r_1, \ldots, r_M)$ whenever person i assigns the number r_i to the object. Second, object a is ranked just as high as b if and only if the value assigned by F to a is at least as great as that assigned by F to b. In practice F is often taken to be the arithmetic mean of the ratings $r_1, \ldots, r_M$ (e.g. Pickering et al., 1973). Observe, however, that this choice of F, in general, produces a non-meaningful ranking of objects, as is shown in the following special case: Suppose $M = 2$ and, for $i = 1, 2$, R_i is person's i representation that is being used for generating ratings, and $R_1(a) = 2$, $R_1(b) = 3$, $R_2(a) = 3$, and $R_2(b) = 1$. Then the arithmetical mean of the ratings for a, 2.5, is greater than that for b, 2, and thus a is ranked above b. However, meaningfulness requires the same order if any other representations of persons 1 and 2 rating scales are used, for example, $10R_1$ and $2R_2$. But for this choice of representations, the arithmetic mean of a, 13, is less than that of b, 16, and thus b is ranked higher than a. It is easy to check that the geometrical mean,

$$F(r_1, \ldots, r_M) = (r_1, \ldots, r_M)^{1/M},$$

gives rise to a meaningful rule for ranking objects. It can be shown under plausible conditions that all other meaningful rules give rise to the same ranking as given by the geometric mean.

More subtle applications of the above concept of meaningfulness have been given, and the interested reader should consult Batchelder (1985) and Roberts (1985) for a wide range of social science examples.

In some contexts, this concept of meaningfulness presents certain technical difficulties that require some modification in the definition of meaningfulness (e.g., see Roberts and Franke, 1976, and Falmagne and Narens, 1983).

THEORIES OF MEANINGFULNESS BASED ON INVARIANCE

The above approach to meaningfulness lacks a serious account as to why it is a good concept of meaningfulness; that is, it lacks a sound theory as to why it should yield correct results. Formulating a serious account is difficult. One tack (Krantz et al., 1971; Luce, 1978; Narens, 1981) is to observe that if meaningfulness expresses valid qualitative relationships, then it must correspond to something purely qualitative, and therefore it should have a purely qualitative description. A long tradition in mathematics for formulating intrinsic qualitative relationships, one going back at least to 19th-century geometry and the famous Erlanger Programme of Felix Klein, is to do so in terms of transformations that leave the situation invariant. Formally, let $\mathscr{X}$ be the given qualitative situation (e.g. a relational structure), and K be a set of isomorphisms of $\mathscr{X}$ into itself. A qualitative relation $R(x_1, \ldots, x_n)$ is said to be *K-invariant* if and only if for each $x_1, \ldots, x_n$ in the domain of $\mathscr{X}$ and each f in K,

$$R(x_1, \ldots, x_n) \quad \text{iff} \quad R[f(x_1), \ldots, f(x_n)].$$

In mathematics, 'intrinsic' has usually been associated with a special type of K-invariance, namely when K is the group (under function composition) of all isomorphisms of $\mathscr{X}$ *onto* itself. These isomorphisms are called *automorphisms*, and this type of invariance is called *automorphism invariance*. The automorphism group has many desirable mathematical properties, including, of course, that the primitive relations that define the qualitative situation are all automorphism invariant. For measurement, it often seems appropriate to use the larger set of all isomorphisms of $\mathscr{X}$ *into* itself, the 1–1 *endomorphisms*. The resulting invariance is called *endomorphism invariance*. One theory of meaningfulness identifies qualitative meaningfulness with automorphism invariance, and another identifies it with endomorphism invariance. Both are based on structure preserving concepts and so relate readily to measurement concerns, since measurement, at least theoretically, is based upon related structure preserving concepts. Although little philosophical justification exists for either of these concepts, they, and especially automorphism meaningfulness, appear to lead to many correct results. For example, automorphism meaningfulness provides a basis of dimensional analysis (as described below). Under these theories, quantitative forms of meaningfulness result from forming images of qualitative meaningful relations by proper means of measurement.

DIMENSIONAL ANALYSIS

In at least four areas of science invariance ideas of meaningfulness have played a fundamental and major role: dimensional analysis in classical physics, the question of meaningful statistical assertions, relativistic physics, and mathematics (especially geometry). Since some applications of the first two have been to economics and other social sciences (de Jong, 1967; Roberts 1985), a brief summary of their main ideas is provided.

Dimensional analysis involves two major concepts: a structure of physical variables – those quantities for which units can be specified – represented as a finite dimensional, multiplicative vector space, and the assumption that any physical law that can be formulated as a relation among variables and constants represented in this space must satisfy an invariance property, called 'dimensional invariance', which is described below. When fully articulated, these two propositions imply Buckingham's (1914) theorem: any such law can be expressed as a function of one or more dimensionless quantities (i.e. real numbers), each of which is a product of powers of some of the variables involved.

Typical applications. Accepting for the moment the correctness of these two major premises of dimensional analysis, consider how they may be used. Without question, the simplest and most widespread use is to check an equation for dimensional consistency. Only quantities with the same dimensions can be added or set equal to one another. An equation failing this property simply cannot describe anything of empirical significance if dimensional invariance is a valid property of physical laws. For a discussion with some economic examples, see Osborne (1978). Most scientists have employed such checks whether or not they are aware of dimensional analysis.

There is, in addition, a much more powerful application of the method. Suppose a process or system is sufficiently well understood so that all of the relevant variables are known. This is a very strong assumption, one we are often unsure of, especially in incompletely developed areas of science. It is, however, met in physical situations when we have a full understanding of the laws at work but are, none the less, unable to solve the resulting equations. In such cases, by using elementary methods of linear algebra, it is possible systematically to develop a set of independent dimensionless combinations of the relevant variables. In that case, Buckingham's theorem tells us that the law is some unspecified function of these dimensionless quantities. If one of the variables of the system is viewed as the dependent one and if it appears in just one of the dimensionless combinations, then it can be solved for. This results in an expression for the dependent variable that is a product of powers of the other variables in that dimensionless combination times an unspecified function of all the other dimensionless quantities. For example, as has been shown in a number of books on the subject, it is easy to derive from dimensional considerations that the lift and drag of an idealized airfoil must be proportional to the square of the velocity, to the density of the air, to the area of the airfoil, to an explicit function of the angle of attack, and to an unknown function of a dimensionless quantity called the 'Reynolds' number'. Many other examples of the effective use of these techniques are routinely found in texts on engineering and applied physics (e.g. Sedov, 1959).

Constructing the dimensional structure. In order to understand the method well enough to see how applicable it may be beyond physics, two issues need to be addressed: where does the vector space representation come from, and why should we postulate that laws are dimensionally invariant? The latter question has attracted more attention than the former, although the concept of dimensional invariance becomes rather transparent once the qualitative underpinnings of the structure of quantities are worked out.

The basic tying together of the dimensions of classical physics are measurement structures involving triples of interrelated attributes. These consist of a conjoint structure, say $\langle A \times P, \succsim \rangle$, that has at least one operation on either A, P, or $A \times P$ such that it together with the ordering induced on that component by $\succsim$ forms a positive concatenation structure with a ratio scale representation. Further, the operation and conjoint structure are interconnected by a qualitative distribution law. For example, if the operation $\circ$ is on A, then it is said to be *distributive* if, for a, b, c, d in A and p, q in P, whenever $(a,p) \sim (c,q)$ and $(b,p) \sim (d,q)$, then $(a \circ b, p) \sim (c \circ d, q)$. (This definition was given independently by Narens and Luce (1976) and Ramsay (1976).) For example, if A represents a set of masses and P a set of velocities and the ordering is by the amount of kinetic energy, then the usual concatenation operation for masses is distributive in this triple. Under plausible solvability and Archimedean conditions, it can be shown (Narens and Luce, 1976; Luce and Narens, 1985; Narens, 1985) that the conjoint ordering has a representation in terms of products of powers of the ratio scale representations of the operations. This fact is reflected in the ordinary pattern of units as products of powers of others, for example the unit of energy is gm^2/t^2. The laws captured by these distributive triples are the most elementary ones that relate several dimensions.

If there are sufficiently many of these distributive triples and if they are sufficiently redundant so that there is a finite basis to the structure, then they can be simultaneously represented numerically as a finite dimensional, multiplicative vector space (Krantz et al., 1971; Luce, 1978; Roberts, 1980). Three major things are used to accomplish this development: a theory of ratio scale representations of concatenation structures, a theory of representations of conjoint structures, and the qualitative concept of an operation being distributive in the conjoint structure. Most traditional accounts attempt to make do only with the first of these elements, usually for the special case of extensive structures, and as a result it is obscure where the rest of the structure comes from.

Relation to meaningfulness. It is plausible that laws formulated within this structure should be meaningful in the sense of invariance under automorphisms of the structure. By a well-known theorem of mathematical logic, it can be shown that this is true of any law that can be defined through (first-order) predicate logic in terms of the primitive relations of the structure. Luce (1978; see also Roberts, 1980) showed that automorphism invariance is equivalent to the following numerical requirement known as dimensional invariance: suppose the numerical law admits a particular combination of values of the relevant variables as a possible configuration of the system in question – that is, these values satisfy the law governing the system. Suppose, further, that an admissible transformation is carried out on these values in the sense that separate admissible transformations are made on each basis variable of the multiplicative vector space and all other variables are transformed as prescribed by that space. Then, according to dimensional invariance, when the combination of values satisfying the law is subject to an admissible dimensional transformation of the sort described, the transformed values also satisfy the law. (Ramsay (1976), in essence, defined 'dimensional invariance' as automorphism invariance, and he showed that distribution of a bisymmetric operation is sufficient to ensure automorphism invariance. He did not, however, show that his conditions imply a multiplicative vector space of units or the product of powers representation. That means that he did not show that his conditions imply the usual concept of dimensional invariance that was described above.)

There seems to be a wide consensus within the physical community that physical laws should be dimensionally invariant, although that community is not very clear – indeed, there is disagreement – as to why this is the case. Attempts have been made to argue for this property on a priori grounds and as a consequence of a concept of physical similarity (Buckingham, 1914; Bridgman, 1931; Causey, 1969; Luce, 1971; Osborne, 1978), but none of these seem as satisfactory as arguing for it in terms of automorphism invariance, which appears to be a more fundamental concept, one that is stated in purely qualitative terms. Thus, it seems to the authors that equivalence to automorphism invariance provides a more rigorous and better foundation for dimensional analysis than do the ones customarily given by physicists and engineers.

Extension beyond classical physics. The current theories for dimensional analysis fail to account adequately for measurements of either relativistic or quantum quantities. For example, at the representational level, relativistic velocity seems to work perfectly well since it continues to be distance divided by time, but because it is a bounded structure and its 'addition' operation is not distributive in the conjoint structure relating distance, velocity, and duration, the existing theorems do not account for why it can be included in the overall dimensional structure. The variables of quantum theory are far more perplexing, and little has been done to incorporate them in such a structure.

A question of natural interest to economists is whether dimensional methods are applicable to their sort of problems. An attempt to show that they are is given in de Jong (1967) and Osborne (1978) (also, see Roberts, 1985). Certainly there are some uses, such as the verification of dimensional consistency of equations. What seems to be lacking in the economic situation, however, is a sufficiently rich set of elementary laws of the type captured as distributive triples in order to set up a full vector space of dimensions like the one found in physics. A similar observation holds for other areas such as psychophysics, which is perhaps as close as any other to creating such a structure. It appears that additional basic work on these measurement questions is needed before it will be possible to bring to bear the full power of these highly useful methods to economics.

Input–output functions. A part of the theory, however, has proved to be promising for both economic and other social science concerns. This involves laws that describe input–output relations among variables of known scale types. In these cases, dimensional invariance simply says that the function relating them must have the following homogeneity property: The effect of admissible scale transformations on the input (independent) variables results in an admissible transformation on the output (dependent) variable. Such a homogeneity condition imposes severe restrictions on the form of the function when all of the input variables are dimensionally independent and even when they are all constrained to have the same dimension (Falmagne and Narens, 1983; Luce, 1959) For example, if there is just one ratio scale input, a ratio scale output, and a strictly increasing output function, then the function must be proportional to a power of the independent variable; if the output is an interval scale, then logarithmic functions can also arise. Such limitations have proved effective in some psychological applications (Luce, 1959; Osborne, 1970, 1976; Iverson and Pavel, 1981; Falmagne, 1985; Roberts, 1985), and they constitute a substantial part of de Jong's (1967) book.

It must be recognized, however, that they really are a presumed application of dimensional analysis in areas that do not have enough structure to justify its use, that is, dimensional invariance is assumed for these special cases without having a theory as to why this should be so. Moreover, one of two very strong assumptions is involved, namely that either all of the independent variables are dimensionally independent or they all have the same dimension.

MEANINGFULNESS AND STATISTICS

Another area of importance to social scientists in which invariance notions are believed to be relevant is the application of statistics to numerical data. The role of measurement considerations in statistics and of invariance under admissible scale transformations was first emphasized by Stevens (1946, 1951); this view quickly became popularized in numerous textbooks, and it resulted in extensive debates in the literature. Continued disagreement exists, mainly created by confusion arising from the following simple facts: measurement scales are characterized by groups of admissible transformations of the real numbers. Statistical distributions exhibit certain invariances under appropriate transformation groups, often the same groups (especially the affine transformations) that arise from measurement considerations. Because of this, some have concluded that the suitability of a statistical test is determined in part by whether or not the measurement and distribution groups are the same. Thus, it is said that one may be able to apply a test, such as a *t*-test, that rests on the Gaussian distribution to ratio or interval scale data, but surely not to ordinal data, because the Gaussian is invariant under the group of affine transformations – which arises in both the ratio and interval case but not in the ordinal one. Neither half of the assertion is correct: first, a significance test should be applied only when its distributional assumptions are met, and they may very well hold for some particular representation of ordinal data. And, second, a specific distributional assumption may well not be met by data arising from ratio scale measurement. For example, reaction times, being times, are measured on a ratio scale, but they are rarely well approximated by a Gaussian distribution.

What is true, however, is that any proposition (hypothesis) that one plans to put to statistical test or to use in estimation had better be meaningful with respect to the scale used for the measurements. In general, it is not meaningful to assert that two means are equal when the quantities are measured by an ordinal scale, because equality of means is not invariant under strictly increasing transformations. Thus, no matter what distribution holds and no matter what test is performed, the result may not be meaningful because the hypothesis is not. In particular, if an hypothesis is about the measurement structure itself, for example that the representation is additive over a concatenation operation, then it is essential that the hypothesis be automorphism invariant and that, moreover, the hypotheses of the statistical test be met without going outside the transformations of the measurement representation.

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See also AXIOMATIC THEORIES; DIMENSIONS OF ECONOMIC QUANTITIES; INTERPERSONAL COMPARISONS OF UTILITY; MEASUREMENT, THEORY OF; TRANSFORMATIONS AND INVARIANCE.

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